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LETTER TO THE EDITOR

The quantum group $SU_q(2)$ and a q-analogue of the boson operators

L C Biedenharn

Department of Physics, Duke University, Durham, NC 27706, USA

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Abstract. A new realisation of the quantum group $SU_q(2)$ is constructed by means of a q-analogue to the Jordan-Schwinger mapping, determining thereby both the complete representation structure and q-analogues to the Wigner and Racah operators. To achieve this realisation, a new elementary object is defined, a q-analogue to the harmonic oscillator. The uncertainty relation for position and momentum in a q-harmonic oscillator is quite unusual.

The quantum group $SU_q(2)$—also denoted $U_q(su(2))$—was first introduced by Sklyanin, and independently by Kulish and Reshetikhin in their work [1] on Yang-Baxter equations; these equations are well known to play a crucial role in classical and quantal integrable systems [2, 3]. Quantum groups themselves were categorised abstractly by Drinfeld [4] in terms of Hopf algebras as the natural algebraic setting for the inverse scattering problem [5] and for conformal field theory [6]. The purpose of the present paper is to develop a new realisation of the quantum group $SU_q(2)$, using a q-analogue of the Jordan-Schwinger mapping [7] which simplifies quite remarkably the algebraic manipulations of the theory. To achieve this realisation we shall construct a new elementary object, a q-analogue of the harmonic oscillator and, using this, a q-analogue of the boson operator calculus. This construction provides a model underlying—and thereby motivating—the q-analogue structure [8] itself; in turn this model leads to some surprising implications that we detail in the concluding discussion.

Let us consider the quantum group, $SU_q(2)$, which is generated algebraically by the operators, J_+, J_- and J_z obeying the Lie bracket (commutator) relations:

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = \frac{q^{J_z} - q^{-J_z}}{q^{1/2} - q^{-1/2}} \tag{2}$$

where q is a real number.

Equations (1) and (2), for general q, no longer define a Lie algebra but an infinite-dimensional associative algebra $U_q(su(2))$, a deformation of the universal enveloping algebra of the Lie algebra su(2). In the limit $q \to 1$, the generators of this algebra, $su_q(2)$, contract to the Lie algebra of SU(2).

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The Jordan–Schwinger approach to SU(2) maps the 2×2 fundamental (spinor) realisation onto a pair of independent (commuting) boson operators:

$$J_+ \to a_1\bar{a}_2 \qquad J_- \to a_2\bar{a}_1 \qquad \text{and} \qquad J_z \to \tfrac{1}{2}(a_1\bar{a}_1 - a_2\bar{a}_2) \tag{3}$$

where $[\bar{a}_i, a_j] = \delta_{ij}$ with all other brackets vanishing.

What is the q-analogue to this construction? To answer this question, we must define a q-analogue to the harmonic oscillator.

We propose this definition: consider the *q-creation operator* a_q, its Hermitian conjugate the *q-destruction operator* $\bar{a}_q$, and the *q-boson vacuum ket*$|0\rangle_q$ defined by $\bar{a}_q|0\rangle_q = 0$. Instead of the Heisenberg (Lie) algebra, we postulate the algebraic relation

$$\bar{a}_q a_q - q^{1/2} a_q\bar{a}_q = q^{-N_q/2} \tag{4}$$

where N_q is the (Hermitian) *number operator*, defined to satisfy

$$[N_q, a_q] = a_q \tag{5}$$

$$[N_q, \bar{a}_q] = -\bar{a}_q. \tag{6}$$

This algebra is a q-analogue generalisation of the Heisenberg algebra, which is itself now easily seen to be the contraction limit $q\to 1$. (Equation (4) was partly inspired by Manin's discussion [8] of non-commutative geometry.)

To construct the state vectors we proceed in the standard way. The unnormalised ket vectors for states of n quanta are clearly of the form $|n\rangle_q = (\text{numerical norm}) \times a_q^n|0\rangle_q$, where a_q^n denotes the nth power of a_q.

To evaluate the norm we iterate equation (4) to find

$$\bar{a}_q a_q^n - q^{n/2} a_q^n \bar{a}_q = \sum_{k=0}^{n-1} q^{k/2} a_q^k q^{-N_q/2} a_q^{n-k-1}. \tag{7}$$

It follows that the n-quanta eigenstates $\{|n\rangle_q\}$ are given by

$$|n\rangle_q = ([n]_q!)^{-1/2} a_q^n |0\rangle_q \tag{8}$$

and are orthonormal.

In equation (8), the q-factorial $[n_q]!$ has for the term $[n]_q$ in the product (with n an integer) the explicit form:

$$[n]_q \equiv q^{(n-1)/2} + q^{(n-3)/2} + \ldots + q^{-(n-1)/2}. \tag{9a}$$

This q-factorial can be related to a q-analogue of Euler's factorial function [9]. More precisely (since q-analogues are not necessarily unique), we have the relation:

$$[n]_q! = q^{-n(n-1)/4}\Gamma_q(n+1) \tag{9b}$$

with $\Gamma_q(x)$ being the q-analogue of the gamma function [9] defined for general (and not just integer) arguments. ($\Gamma_q(x)\to\Gamma(x)$ for $q\to 1$; as usual, $\Gamma_q(1)\equiv 1 = [0]_q!$).

The normalisation factors† for the eigenstates are invariant to the symmetry $q\to q^{-1}$. It is useful to note that *the number operator* N_q *is not* $a_q\bar{a}_q$.

It is now an easy matter to define the q-analogue to the Jordan–Schwinger map.

To realise the Lie algebra of the generators of $SU_q(2)$, we define a pair of mutually commuting q-harmonic oscillator systems: a_{iq} and $\bar{a}_{iq}$ with $i = 1, 2$. Then we have:

$$J_+ = a_{1q}\bar{a}_{2q} \tag{10}$$

$$J_- = (J_+)^+ = a_{2q}\bar{a}_{1q} \tag{11}$$

† There are, in fact, several possible q-analogues for the q-harmonic oscillator structure. The invariance of $[n]_q$ in equation (8) under $q\to q^{-1}$ makes the choice unique. This invariance stems from the related invariance of $su_q(2)$.

and

$$J_z = \tfrac{1}{2}(N_{1q} - N_{2q}). \tag{12}$$

The eigenstates $|j, m\rangle_q$ are now q-analogues of the familiar quantal angular momentum states:

$$|j, m\rangle_q \equiv ([j+m]_q![j-m]_q!)^{-1/2} a_{1q}^{j+m} a_{2q}^{j-m} |0\rangle_q. \tag{13}$$

One easily verifies that

$$J_\pm |j, m\rangle_q = ([j \mp m]_q [j \pm m + 1]_q)^{1/2} [j, m \pm 1\rangle_q \tag{14}$$

and that

$$J_z |j, m\rangle_q = m|j, m\rangle_q. \tag{15}$$

The value of j is determined by the weight of the highest-weight state.

To verify the defining commutation relations, (1) and (2), we see, using (14) and (15) and after some algebraic manipulation, that

$$\begin{aligned}[][J_+, J_-]|j, m\rangle_q &= ([j+m]_q[j-m+1]_q - [j-m]_q[j+m+1]_q)|j, m\rangle_q \\ &= \left(\frac{q^{J_z} - q^{-J_z}}{q^{1/2} - q^{-1/2}}\right)|j, m\rangle_q \end{aligned} \tag{16}$$

$$[J_z, J_\pm]|j,\rangle_q = \pm J_\pm |jm\rangle_q. \tag{17}$$

This verifies the defining algebraic relations of $su_q(2)$. It is essential to note, however, that this q-analogue of the Jordan–Schwinger mapping verifies these defining relations *only on ket vectors that terminate with the q-vacuum ket.* Unlike the usual result, the commutation relations do not close abstractly†.

The realisation for $su_q(2)$ defined by (10)–(12), and the set of eigenkets $\{|j, m\rangle_q\}$ given in (13) define finite-dimensional unitary irreps of $su_q(2)$ for every $j = 0, \frac{1}{2}, 1, \ldots$ with m running by integer steps over the range $j \geqslant m \geqslant -j$. It is useful to note that for $j = \frac{1}{2}$, the $su_q(2)$ generators are exactly the Pauli matrices, $J_i^{(j=1/2)} = \frac{1}{2}\sigma_i$.

It is easily seen that if we have two distinct (commuting) realisations of $su_q(2)$, say $J_i(1)$ and $J_i(2)$, then the sum: $J_i = J_i(1) + J_i(2)$, does not in general obey equation (2). *Thus the usual technique of 'adding angular momenta' fails* and a q-analogue to the Wigner–Clebsch–Gordan (WCG) coefficients must be defined in a logically different way. As developed in [7], an alternative characterisation, equivalent for su(2) but required for $su_q(2)$, defines WCG coefficients as matrix elements of *unit tensor operators.* The basic building block is the $j = \frac{1}{2}$ tensor operator, from which all other q-WCG operators for $SU_q(2)$ may be constructed.

Let us illustrate this concept by calculating the $j = \frac{1}{2}$ q-WCG coefficients. The operator a_{1q} corresponds to a tensor operator which induces the changes: $j \to j + \frac{1}{2}$ and $m \to m + \frac{1}{2}$ on a generic basis vector, so that the generic matrix element of a_{1q} is an *unnormalised* q-WCG coefficient:

$$(\Delta j = \tfrac{1}{2}, \Delta m = \tfrac{1}{2}) \qquad a_{1q} \to ([j+1+m]_q)^{1/2}. \tag{18a}$$

Similarly:

$$(\Delta j = \tfrac{1}{2}, \Delta m = -\tfrac{1}{2}) \qquad a_{2q} \to ([j+1-m]_q)^{1/2} \tag{18b}$$

$$(\Delta j = -\tfrac{1}{2}, \Delta m = \tfrac{1}{2}) \qquad \bar{a}_{2q} \to ([j-m]_q)^{1/2} \tag{18c}$$

$$(\Delta j = -\tfrac{1}{2}, \Delta m = -\tfrac{1}{2}) \qquad -\bar{a}_{1q} \to -([j+m]_q)^{1/2} \tag{18d}$$

† This situation is not new and occurs already in Dirac's theory of constraints. See also [10].

where (j, m) refer to the initial state $|jm\rangle$. These four matrix elements, *suitably normalised*, constitute the $j=\frac{1}{2}$ q-WCG coefficients.

It is clear from this example that the complete calculation of the general q-WCG coefficient in $SU_q(2)$ is a direct q-boson transcription of the boson calculations given in [7].

The product of unit tensor operators is again a tensor operator, which can be split into unit tensor parts by the q-analogue WCG coefficients. This operator algebraic process [7] allows us to define the q-Racah coefficients as matrix elements of the *invariant operator* $(A \cdot B \times C)$ where A, B, C and q-WCG operators and the two multiplications: $(\cdot = \text{scalar})$ and $(\times)$ are defined by q-WCG coefficients (appropriately chosen to produce an overall invariant).

So far we have simply shown an elegant and suggestive way to obtain certain known [11, 12]—as well as new—results in $su_q(2)$. The really interesting questions are those concerning the physics implied by the q-harmonic oscillator structure. To determine this we proceed by analogy. Let us *define* the q-momentum (P_q) and q-position (Q_q) operators directly from the q-boson operators a_q and $\bar{a}_q$ introduced in equations (4)-(6). That is,

$$P_q \equiv \mathrm{i}\sqrt{m\hbar\omega/2}(a_q - \bar{a}_q) \tag{19}$$

and

$$Q_q \equiv \sqrt{\hbar/2m\omega}(a_q + \bar{a}_q). \tag{20}$$

Using these operators, we define, again by analogy, the q-analogue harmonic oscillator Hamiltonian to be: $\mathcal{H}_q \equiv (P_q)^2/2m + \frac{1}{2}m\omega^2(Q_q)^2$, so that

$$\mathcal{H}_q = \tfrac{1}{2}\hbar\omega(\bar{a}_q a_q + a_q \bar{a}_q). \tag{21}$$

This q-Hamiltonian operator $\mathcal{H}_q$ is diagonal on the eigenstates $|n\rangle_q$ and has the eigenvalues:

$$\mathcal{H}_q \to E_q(n) = \tfrac{1}{2}\hbar\omega([n+1]_q + [n]_q). \tag{22}$$

We see immediately that the energy levels are no longer uniformly spaced (for $q \neq 1$).

Consider next the uncertainty relation for the q-position and q-momentum, i.e. the commutator $\mathrm{i}[P_q, Q_q] = [\bar{a}, a]$. This operator is diagonal (on the eigenstates $\{|n\rangle_q\}$ and has the eigenvalues:

$$\mathrm{i}[P_q, Q_q] \to \hbar([n+1]_q - [n]_q) = \hbar\frac{\cosh(\frac{1}{4}(2n+1)\ln q)}{\cosh(\frac{1}{4}\ln q)}. \tag{23}$$

One sees that the uncertainty is *minimal* (and independent of n) only in the limit $q \to 1$; the uncertainty *increases* with n for $q \neq 1$. This shows that any attempt to measure position accurately in a q-harmonic oscillator will necessarily involve large energies and a corresponding characteristic *increase* in the intrinsic uncertainty of equation (20).

Spontaneous and stimulated emission for q-boson fields originates in the q-boson factors: $a_q \to ([n+1]_q)^{1/2}$ for creation as opposed to $\bar{a}_q \to ([n]_q)^{1/2}$ for stimulated absorption. The difference between these factors is also given by equation (23), which shows that the spontaneous emission probability of a q-harmonic oscillator *increases* for large occupation numbers (n) and is independent of n (and unity) only for $q \to 1$, the standard boson result†. Let us note that it is possible to define coherent states $\{|\alpha\rangle_q\}$ for q-harmonic oscillators, using the definition:

$$\bar{a}_q|\alpha\rangle_q \equiv \alpha|\alpha\rangle q \tag{24a}$$

with

$$|\alpha\rangle_q = \exp_q(-\tfrac{1}{2}|\alpha|^2) \sum_{k=0}^{\infty} \frac{\alpha^k a_q^k}{([k]_q!)} |0\rangle_q \tag{24b}$$

where $\exp_q(x)$ is the q-analogue of the exponential function. The energy distribution in such a q-coherent state is now a q-analogue Poisson distribution.

The physical reality of these deviations from standard quantum mechanics, which originate in the basic characteristics of q-harmonic oscillators, can only be a matter of speculation at present, and certainly any such deviations can be expected to be extremely small at ordinary energies. At the Planck scale, however, energies are enormously large and the implied deviations could be significant (and possibly even helpful in field theory). The solvable lattice models of statistical mechanics and the field theoretic examples of quantum groups are intimately connected with such q-analogue structures and are certainly well founded physically; it is this known, and important, fact that provides the motivation for taking q-harmonic oscillators and their strange properties seriously.

Let us remark that the q-boson methods discussed above have been shown to extend directly to all $SU_q(n)$ and, very probably, also to q-analogues of all the classical groups. This extension requires a new concept, a q-analogue to antisymmetrised multiboson operators (determinantal bosons), which we have shown explicitly validates the pattern calculus rules [13] (replacing factorials by $[n]_q!$) for these q-analogue operator structures. This fundamental result in turn yields explicitly the complete representation structure for all $su_q(n)$, and q-analogues for the WCG and Racah coefficients, to the extent that these structures are known to be canonically defined in SU(n) itself. A detailed discussion of these results is in preparation.

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Note added in proof. After submission of this paper, a preprint entitled, 'On q-Analogues of the Quantum Harmonic Oscillator and the Quantum Groups SU(2)' by A J Macfarlane was received. This preprint develops some, but not all, of the results given above.

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† We remark, however, that because of the invariance $q \to q^{-1}$, it is possible to obtain Hermitian generators (J_x, J_y, J_z) *not only for q real,* as above, *but also for q complex and of modulus unity.* This modifies equation (23), replacing the hyperbolic functions by cosines, so that we are no longer restricted to the RHS of equation (23) being larger than unity. This possibility may be of physical importance.

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